AIAA JOURNAL
Vol. 43, No. 10, October 2005

Optimal Reciprocalization of Measured Displacements

Menahem Baruch*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

A technique is described by which measured displacements of a linear structure, which usually do not satisfy the theoretical requirement of reciprocity, are forced to satisfy this condition in an optimal way. The corrected displacements are closest to the measured ones in a Euclidean sense. Techniques for computing the corrected displacements are presented and discussed. It is shown that the method proposed here is superior and sometimes equivalent to the previously proposed one.

Nomenclature

A, $a_{\alpha\beta}$	=	positive definite symmetric matrix
B, B_{ij}	=	general matrix
Euc	=	Euclidean norm of a matrix
F	=	load matrix
F_K	=	Kronecker load matrix
f, f_{ij}	=	positive definite symmetric matrix
$\boldsymbol{g}$	=	vectorized γ matrix
H, H_α	=	matrix and its derivatives
K	=	updated stiffness matrix
$\tilde{K}$	=	analytical stiffness matrix
k	=	number of independent elements
m	=	number of loadings
Nc, Ns	=	Frobenius norms
n	=	degrees of freedom
$\boldsymbol{p}$, $\boldsymbol{p}_\alpha$	=	parametric vector
$\boldsymbol{q}$, $\boldsymbol{q}_\alpha$	=	vectorized β matrix
$\boldsymbol{r}$, $\boldsymbol{r}_\alpha$	=	vector
S	=	symmetric matrix
S_c, S_{cij}	=	conditional symmetric matrix
S_K	=	symmetric Kronecker matrix
T, T_{ij}	=	measured displacements
X	=	corrected displacement matrix
X_c, X_{cij}	=	conditionally corrected displacement matrix
X_K	=	corrected Kronecker displacement matrix
Y	=	theoretical displacement matrix
β, β_K	=	Lagrange multiplier matrices
γ	=	skew-symmetric matrix
η_c	=	Lagrange multiplier matrix
λ_c, λ_{cij}	=	Lagrange multiplier matrix
Φ	=	positive definite symmetric matrix
ϕ_1, ϕ_2, ϕ_K, ϕ	=	Lagrange functions
Ψ	=	flexibility matrix

Introduction

IT is well known that the displacements of a given structure that are determined experimentally usually do not satisfy the requirement of reciprocity, that is, they do not fulfill the Maxwell–Betti reciprocal theorem (see Ref. 1). A method to satisfy this requirement was previously proposed.[2–5] However, it is found that by using the method proposed here one obtains corrected displacements that are closer or equally close to the measured displacements in a Euclidean sense than the ones obtained by the previously proposed method. Measured displacements that fulfill the reciprocal theorem can be used to identify the stiffness matrix, the dynamic matrix, or the damping matrix of a structure. The Maxwell–Betti reciprocal theorem has been used also to obtain integral equations for nondestructive determination of buckling.[6,7] A somewhat similar approach to achieve optimal weighted orthogonalization of measured modes was proposed earlier.[8]

The previously proposed solution[2–5] of the reciprocalization of the measured displacements was a conditional approach. This approach needed additional requirements. In 1997 Lallement[9] proposed to try a self-sufficient approach, an approach that does not need additional conditions. This approach is somewhat more complicated. However, as will be shown, it can be done. In any case, it will be demonstrated that the self-sufficient approach is more appropriate than the conditional one.

Problem

A linear structure must fulfill the Maxwell–Betti reciprocal theorem. This is true for static or dynamic environments.[1–5] This theorem expressed in matrix form reads

$$F^t Y = Y^t F \tag{1}$$

where $F\,(n \times m)$ is the load matrix and $Y\,(n \times m)$ is the displacement matrix caused by the load matrix applied on the structure. Usually the measured displacements $T\,(n \times m)$, due to different mistakes, do not fulfill the reciprocal theorem. However for further dynamic or static computations the reciprocal theorem must be fulfilled. The problem is to correct the displacement matrix in an optimal way in which the reciprocal theorem is fulfilled and the corrected displacement matrix $X\,(n \times m)$ is as close as possible to the measured displacement matrix. Here the closeness of two matrices will be ascertained in a Euclidean sense.

Conditional Approach

Applying the conditional approach,[2–5,10] the following elegant close-form solution was obtained:

$$X_c = T + 1/2F(F^tF)^{-1}(T^tF - F^tT) \tag{2}$$

where X_c are the conditionally obtained displacements that fulfil the Maxwel–Betti reciprocal theorem. It is clear that $F\,(n \times m)$ must be of rank (m). To obtain Eq. (2) the additional condition was inserted in a concealed way.[2–5] To clarify the meaning of the additional condition and to compare the conditional with the self-sufficient approach we will apply the conditional approach in a way somewhat different from the one applied earlier.[2–5]

The basic requirements are that the matrix F^tX_c has to be symmetric and the corrected displacement matrix X_c as close as possible to the measured displacement matrix T in a Euclidean sense. Hence, find a matrix X_c that minimizes the Euclidean norm $\|X_c - T\|$ and makes the matrix F^tX_c symmetric:

$$\phi_1 = 1/2\|X_c - T\| + \lambda_c\left|F^tX - S_c\right| = 1/2(X_{cij} - T_{ij})^2 + \lambda_{cik}(F_{ji}X_{cjk} - S_{cik}) \tag{3}$$

Received 3 December 2002; accepted for publication 20 May 2004.

*Professor Emeritus, Faculty of Aerospace Engineering.

where the Einstein rule for repeated indices was applied. Here S_c is a symmetric matrix assumed known but undefined. The partial differentiation of Eq. (3) with respect to X_{cij}, where the results are equated to zero, yields equations that X_{cij} have to satisfy when ϕ_1 is minimal. Hence,

$$\frac{\partial \phi_1}{\partial X_c} = X_c - T + F\lambda_c = 0 \tag{4}$$

Multiplication by F^t yields

$$\lambda_c = (F^t F)^{-1}\left(F^t T - S_c\right) \tag{5}$$

$$X_c = T + F(F^t F)^{-1}\left(S_c - F^t T\right) \tag{6}$$

In Eq. (6) S_c is undefined, and this is exactly the needed additional condition.

If $T^t F$ happened to be symmetric no correction for X_c would be needed. Hence, it is logical to look for a symmetric matrix that is the closest to $T^t F$. This can be achieved by minimization of the following Lagrange function:

$$\phi_2 = 1/2\left\| S_c - T^t F \right\| + 1/2\eta_c\left| S_c - S_c^t \right| \tag{7}$$

where due to the nonsymmetrical constraint,

$$\eta_c^t = -\eta_c \tag{8}$$

From Eq. (8) one obtains

$$\frac{\partial \phi_2}{\partial S_c} = S_c - T^t F + \eta_c = 0 \tag{9}$$

The transpose of Eq. (9) will give

$$S_c - F^t T - \eta_c = 0 \tag{10}$$

Equations (9) and (10) yield

$$S_c = 1/2(T^t F + F^t T) \tag{11}$$

By substitution of Eq. (11) into Eq. (6) one obtains Eq. (2).

Self-Sufficient Approach

Load Matrix Is Kronecker Matrix

Before going to the general case we will try to solve the case when[9] the load matrix is the Kronecker matrix and then

$$F_K^t F_K = I \tag{12}$$

The Lagrange function, to be minimized, is given as follows:

$$\phi_K = 1/2\|X_K - T\| + 1/2\beta_K\left|F_K^t X_K - X_K^t F_K\right| \tag{13}$$

here again,

$$\beta_K^t = -\beta_K \tag{14}$$

By differentiation of ϕ_K with respect to X_K and equating the result to zero, one obtains

$$X_K = T - F_K \beta_K \tag{15}$$

$$F_K^t X_K = F_K^t T - \beta_K, \qquad X_K^t F_K = T^t F_K + \beta_K \tag{16}$$

Equations (16) yield

$$S_K = F_K^t X_K = X_K^t F_K = 1/2\left(F_K^t T_K + T_K^t F_K\right) \tag{17}$$

$$\beta_K = 1/2\left(F_K^t T - T^t F_K\right) \tag{18}$$

By substitution of Eq. (18) one obtains

$$X_K = T + 1/2F_K\left(T^t F_K - F_K^t T\right) \tag{19}$$

Some interesting observations have to be indicated: The symmetric matrix S_K was obtained without any additional conditions, and when the load matrix is the Kronecker matrix, the symmetric matrix is equal to the symmetric matrix obtained as an additional condition, Eq. (11). Equation (19) can be obtained from Eq. (2) by simply changing F with F_K. All of this looks like a justification of Eq. (2)! Is it?

General Case

In the conditional approach an elegant solution was obtained. However, a better, although not so elegant, solution can be obtained by using a self-sufficient approach. In this approach the only requirement of the load matrix F is that it be of rank (m).

Hence,

$$F^t F = f \tag{20}$$

where $f(m \times m)$ is a positive definite symmetric matrix. The Lagrange function now reads

$$\phi = 1/2\|X - T\| + 1/2\beta|F^t X - X^t F| \tag{21}$$

and again, the Lagrange matrix β is a skew-symmetric matrix,

$$\beta^t = -\beta \tag{22}$$

By equating to zero the first derivative of ϕ in respect to X, one obtains

$$X = T - F\beta \tag{23}$$

$$F^t X = F^t T - f\beta, \qquad X^t F = T^t F + \beta f \tag{24}$$

Equations (24) yield

$$S = F^t X = X^t F = 1/2(F^t T + T^t F + \beta f - f\beta)$$

$$f\beta + \beta f = F^t T - T^t F = \gamma \tag{25}$$

where γ is a skew-symmetric matrix. Note that the symmetric matrix S was obtained directly from Eqs. (24) without any assumptions and usually is different from S_c [Eq. (11)]. It is clear that no additional conditions were applied, and hence the self-sufficient approach provides the optimal reciprocalization of the measured displacements. What remains is to calculate β.

Techniques for Calculation of β

Techniques for calculation of β from the equation

$$f\beta + \beta f = \gamma \tag{26}$$

are as follows.

A closed-form solution of Eq. (26) is not available, and clearly this complicates the self-sufficient method. We propose here two numerical techniques for its solution.

1) *Direct vectorization*. In this technique the parts of Eq. (26) are vectorized. The number of the independent elements of β and γ is

$$k = m(m-1)/2 \tag{27}$$

By vectorization of β and γ one obtains

$$\Phi \boldsymbol{q} = \boldsymbol{g} \tag{28}$$

where $\Phi(k \times k)$ is a symmetric matrix obtained from the elements of the symmetric positive definite matrix f and $\boldsymbol{q}(k)$, $\boldsymbol{g}(k)$ are vectors composed from the independent elements of β and γ, respectively.

For example, if $m = 3$, the number of unknowns is 3 and the vectorization will read

$$\begin{bmatrix} (f_{11}+f_{22}), f_{23}, -f_{13} \\ f_{23}, (f_{11}+f_{33}), f_{12} \\ -f_{13}, f_{12}, (f_{22}+f_{33}) \end{bmatrix} \begin{Bmatrix} \beta_{12} \\ \beta_{13} \\ \beta_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \tag{29}$$

If the matrix f were equal to the unit matrix $I(m \times m)$ (see preceding section) the matrix Φ would be equal to twice the unit matrix $I(k \times k)$. Hence, due to continuity Φ has to be a symmetric positive definite matrix.

2) *Minimization of the Euclidean norm of the matrix* $(H = f\beta + \beta f - \gamma)$. For the convenience of the reader we will briefly show here the minimization process with respect to the parameters $\boldsymbol{p}_\alpha (\alpha = 1 - k)$ of the Euclidean norm of a general matrix H (for more details see Ref. 11). It seems that this technique is less cumbersome than the previous one.

The Euclidean norm Euc of the matrix H will be defined as follows:

$$\text{Euc} = \tfrac{1}{2}\text{trace}(H^t H) \tag{30}$$

Usually the matrix H is known or can be calculated for some initial values of the parameters. Taking only the linear part of the Maclaurin–Taylor series one obtains

$$H(\boldsymbol{p}+\boldsymbol{q}) \approx H(\boldsymbol{p}) + \frac{\partial H}{\partial \boldsymbol{p}_\alpha}\boldsymbol{q}_\alpha = H(\boldsymbol{p}) + H_\alpha \boldsymbol{q}_\alpha \tag{31}$$

where the Einstein rule of summing quantities with repeated indexes was applied.

Now,

$$\text{Euc} = \tfrac{1}{2}\text{trace}\left\{[H(\boldsymbol{p}) + H_\chi \boldsymbol{q}_\chi]^t [H(\boldsymbol{p}) + H_\beta \boldsymbol{q}_\beta]\right\} \tag{32}$$

The minimum of Euc will be found by

$$\frac{\partial \text{Euc}}{\partial \boldsymbol{q}_\alpha} = \text{trace}\left[H_\alpha^t (H + H_\beta \boldsymbol{q}_\beta)\right] = 0 \tag{33}$$

Hence,

$$A\boldsymbol{q} = \boldsymbol{r} \tag{34}$$

Following the continuity reasons stated in the preceding section, the matrix $A(k \times k)$ has to be also a symmetric positive definite matrix. The elements $\alpha_{\alpha\beta}$ of A are given as follows:

$$\alpha_{\alpha\beta} = \text{trace}\left(H_\alpha^t H_\beta\right) \tag{35}$$

$$\boldsymbol{r}_\alpha = -\text{trace}\left(H_\alpha^t H\right) \tag{36}$$

The variables $\boldsymbol{q}_\alpha$ that minimize the Euclidean norm, Eq. (30), are calculated from Eq. (34). Here the matrix H is a linear function of the parameters β_{ij}. In this case Eq. (31) represents the exact Maclaurin–Taylor series and iterations are not needed. Hence,

$$\boldsymbol{p}_\alpha = \boldsymbol{p}_\alpha(0) + \boldsymbol{q}_\alpha \tag{37}$$

where $\boldsymbol{p}_\alpha(0) = 0$ represent the initial values of the parameters.

Of course, after the calculation of β by one of the proposed techniques, the corrected displacements X can be obtained by using Eq. (23).

Numerical Example

To generate displacements the following positive definite symmetric matrix was applied:

$$\Psi = \begin{bmatrix} 7 & 5 & 3 & 1 & 0 \\ 5 & 6 & 4 & 2 & 1 \\ 3 & 4 & 8 & 4 & 2 \\ 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 2 & 4 & 9 \end{bmatrix} \tag{38}$$

The matrix $\Psi(5 \times 5)$ represents here the flexibility matrix of some structure. The load matrix $F(5 \times 4)$ was chosen quiet arbitrarily to be

$$F = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \tag{39}$$

The rank of F is of course four as required. The "measured" displacements $T(5 \times 4)$ were obtained by adding to the displacements $Y = \Psi F$ randomly obtained ±1%–±5% "errors." The errors were obtained by using the uniformly random generator of Matlab.[12] The corrected displacements were calculated by applying the conditional approach and the self-sufficient one. To compare the closeness of the corrected displacements X_c and X to T, the Frobenius norms[12] Nc and Ns of $(X_c - T)$ and $(X - T)$, respectively, were calculated for a typical case and plotted in Fig. 1. The Frobenius norm of a matrix B is defined as follows:

$$\text{Frobenius norm}(B) = \sqrt{B_{ij}^2} \tag{40}$$

where the Einstein rule for repeated indices was applied again.

As explained earlier, the measured displacements T were obtained by using a random generator. Hence, every calculation gave different results. Many calculations were performed. The important result is that, as expected, the distance Ns between X and T obtained by employing the self-sufficient method proposed here was always less than the distance Nc between X_c and T obtained by employing the conditional method.

The calculations connected with Fig. 1 were repeated 100 times and averaged. The results are given in Fig. 2.

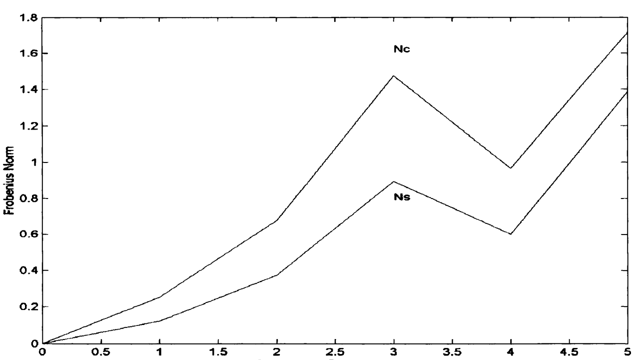

Fig. 1 Typical case: comparison between the conditional and the self-sufficient methods.

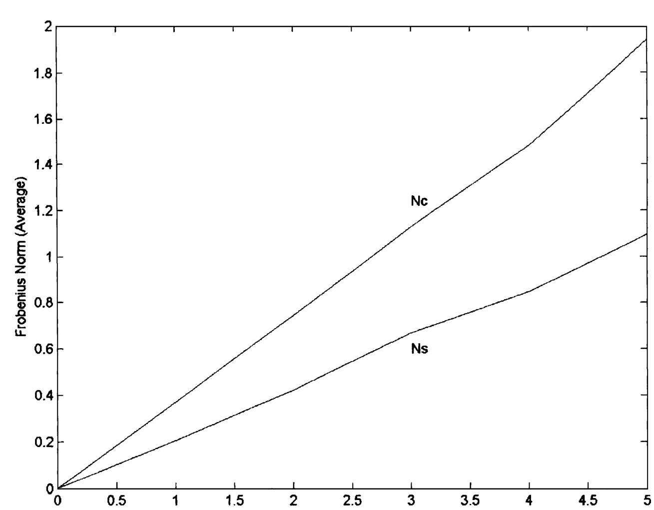

Fig. 2 Averaged results from 100 runs: comparison between the conditional and the self-sufficient methods.

Figure 2 shows clearly the advantage of the self-sufficient method over the conditional one.

Discussion

The reciprocalized measured displacements are needed, for example, to identify the stiffness matrix by utilizing statically measured displacements. A closed form solution to update the stiffness matrix can be found in Refs. 2 and 3:

$$K = \tilde{K} - X(X^t X)^{-1}(X^t \tilde{K} - F^t)[I - 1/2X(X^t X)^{-1}X^t] - [I - 1/2X(X^t X)^{-1}X^t](\tilde{K}X - F)(X^t X)^{-1}X^t \tag{41}$$

Another application of the reciprocalized displacements can be found in Ref. 5, where they are used to identify the damping matrix.

Conclusions

It was shown that the method proposed here for reciprocalization of measured displacements is superior to the previously proposed one.

References

[1]Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, McGraw–Hill, New York, 1956, p. 391.

[2]Baruch, M., "Methods of Reference Basis for Identification of Linear Dynamic Structures," *Proceedings of the AIAA/ASME/ASCE/AHS 23rd Structures, Structural Dynamics, and Material Conference*, AIAA, Washington, DC, 1982, pp. 557–563.

[3]Baruch, M., "Methods of Reference Basis for Identification of Linear Dynamic Structures," *AIAA Journal*, Vol. 22, No. 4, 1984, pp. 561–564.

[4]Baruch, M., Stavrinidis, C., and Fissete, E., "Identification of Fixedly Tested Free Space Structures," ESA, Rept. Y/SB/963.89/jb, Noordwijk, The Netherlands, Aug. 1989.

[5]Baruch, M., "Identification of the Damping Matrix," *Contemporary Research in Engineering Mechanics*, edited by G. A. Kardomateas and V. Birman, American Society of Mechanical Engineers, New York, 2001, pp. 141–160; also American Society of Mechanical Engineers, IMECE2001/AD-23763, Nov. 2001.

[6]Baruch, M., "Integral Equations for Nondestructive Determination of Buckling Loads for Elastic Bars and Plates," *Israel Journal of Technology*, Vol. 11, No. 1–2, 1973, pp. 1–8.

[7]Baruch, M., "Nondestructive Determination of Buckling for Plates and Bars Including Shear," *AIAA Journal*, Vol. 40, No. 11, 2002, pp. 2354–2359.

[8]Baruch, M., and Bar-Itzhack, I. Y., "Optimal Weighted Orthogonalization of Measured Modes," *AIAA Journal*, Vol. 16, No. 4, 1978, pp. 346–351.

[9]Lallement, G., "A Letter to Professor Menahem Baruch," Applied Mechanics Lab. Raymond Chaleat, Université de Frenche-Comté, Besancon, France, Aug. 1997.

[10]Baruch, M., "A Letter to Professor Gerard Lallement," Technion—Israel Inst. of Technology, Haifa, Israel, Aug. 1997.

[11]Baruch, M., "Parametric Methods of Reference Basic for the Identification of Linear Dynamic Structures," *Proceedings, 35th Israel Annual Conference on Aerospace Sciences*, Technion—Israel Inst. of Technology, Haifa, Israel, 1995, pp. 95–105.

[12]MATLAB—Mathematical Computing Software, The Math Works, Natick, MA, 2002.

C. Pierre
Associate Editor